

# Math 451: Introduction to General Topology

## Lecture 21

We now discuss non-permanence of properties under products.

Example (product of two normal isn't normal). Recall that the Sorgenfrey plane  $S \times S$ , where  $S := \mathbb{R}$  is the Sorgenfrey line whose top. is gen. by  $(a, b)$ , is not normal (optical HW), yet the Sorgenfrey line  $S$  is normal because it is regular and Lindelöf.

First ctbls (hence metrizable) are not closed under unctbl products, more precisely:

Prop. Let  $X := \prod_{i \in I} X_i$  be a product of unctblly many nontrivial  $T_1$  top. spaces  $X_i$ . Then  $X$  is isn't 1<sup>st</sup> ctbl; in fact, no point has a ctbl neighbourhood basis.

Proof. Let  $x \in X$  and suppose towards a contradiction that  $x$  has a ctbl neigh. basis  $\{U_n\}_{n \in \mathbb{N}}$ . Since for each  $n \in \mathbb{N}$ ,  $\exists$  finite-base cylinder  $C_n$  s.t.  $x \in C_n \subseteq U_n$ ,  $\{C_n\}_{n \in \mathbb{N}}$  is also a neighbourhood basis of  $x$ . Each cylinder  $C_n$  has a finite base  $I_n \subseteq I$ . Then  $I' := \bigcup_{n \in \mathbb{N}} I_n$  is still ctbl, while  $I$  is unctbl, so  $\exists i_0 \in I \setminus I'$ . Let  $a \in X_{i_0}$  be distinct from  $x(i_0)$ . Since  $X_{i_0}$  is  $T_1$ ,  $\exists$  open  $U_{i_0} \subseteq X_{i_0}$  with  $x(i_0) \in U_{i_0}$  but  $a \notin U_{i_0}$ . Then  $[i_0 \mapsto U_{i_0}]$  is an open neighbourhood of  $x$ , so  $\exists n$  s.t.  $C_n \subseteq [i_0 \mapsto U_{i_0}]$ . But the point  $y \in X$  defined by  $y(i) := \begin{cases} x(i) & i \neq i_0 \\ a & i = i_0 \end{cases}$  belongs to  $C_n$  since  $i_0 \notin I_n$ , but is not in  $[i_0 \mapsto U_{i_0}]$ , contradicting  $C_n \subseteq [i_0 \mapsto U_{i_0}]$ .  $\square$

Compactness (= topological finiteness).

Def. A top. space is called compact if every open cover admits a finite subcover.

Thus, this is a strengthening of Lindelöf property.


Rephrasing compactness via closed sets. A top. space  $X$  is compact  $\Leftrightarrow$  every collection  $\mathcal{C}$  of closed sets with the finite intersection property has a nonempty intersection, i.e.

$$\bigcap \mathcal{C} := \bigcap_{C \in \mathcal{C}} C \neq \emptyset.$$

We say that a collection  $\mathcal{C}$  of sets satisfies the finite intersection property (FIP) if every finite  $\mathcal{C}_0 \subseteq \mathcal{C}$  has a nonempty intersection, i.e.  $\bigcap_{C \in \mathcal{C}_0} C \neq \emptyset$ .

Proof.  $\Rightarrow$  Let's prove the contrapositive: suppose there is a collection  $\mathcal{C}$  of closed sets with FIP but  $\bigcap \mathcal{C} = \emptyset$ . Then let  $\mathcal{U} := \{X \setminus C : C \in \mathcal{C}\}$  (called the dual collection). Then  $\mathcal{U}$  is an open cover of  $X$  because  $\emptyset = \bigcap \mathcal{C} \Rightarrow$  (via complements)  $X = \bigcup \mathcal{U}$ , by deMorgan. Yet  $\mathcal{U}$  has no finite subcover: for any finite  $\mathcal{U}_0 \subseteq \mathcal{U}$ ,  $\emptyset \neq \bigcap_{U \in \mathcal{U}_0} U^c$  because of FIP, hence (by deMorgan),  $X \neq \bigcup_{U \in \mathcal{U}_0} U$ .

$\Leftarrow$ . If  $\mathcal{U}$  is an open cover of  $X$ , then the dual of  $\mathcal{U}$ , namely,  $\mathcal{C} := \{X \setminus U : U \in \mathcal{U}\}$  is a collection of closed sets with  $\emptyset$  intersection because  $X = \bigcup \mathcal{U}$  implies  $\emptyset = \bigcap \mathcal{C}$ . Thus,  $\mathcal{C}$  doesn't have the FIP, i.e.  $\exists$  finite  $\mathcal{C}_0 \subseteq \mathcal{C}$  with  $\bigcap \mathcal{C}_0 = \emptyset$ , so (by deMorgan),  $\bigcup_{C \in \mathcal{C}_0} C^c = X$ , so  $\mathcal{U}_0 := \{C^c : C \in \mathcal{C}_0\}$  is a finite subcover.  $\square$

Def. Let  $\mathcal{U}$  be a cover of a set  $X$ . Another cover  $\mathcal{V}$  of  $X$  is called a refinement of  $\mathcal{U}$  if every  $V \in \mathcal{V}$  is contained in some  $U \in \mathcal{U}$ . 

Prop. For every open cover  $\mathcal{U}$  of a top. space  $X$ , every basis  $\mathcal{B}$  of  $X$  contains an open cover of  $X$  refining  $\mathcal{U}$ .

Proof. For each  $x \in X$   $\exists U \in \mathcal{U}$  covering  $x$ , i.e.  $x \in U$ , so  $\exists B_x \in \mathcal{B}$  with  $x \in B_x \subseteq U$ . Then  $\mathcal{V} := \{B_x\}_{x \in X}$  is an open cover of  $X$  and refines  $\mathcal{U}$  by definition.  $\square$

Obs. Let  $X$  be a top. space. (FIP)

(a)  $X$  is compact  $\Leftrightarrow$  every open cover admits a finite refinement.

(b)  $X$  is Lindelöf  $\Leftrightarrow$  every open cover admits a countable refinement.

Cor. In the def of compactness it suffices to consider open covers consisting of sets in some basis. More precisely, a top. space  $X$  is compact  $\Leftrightarrow$  for every basis  $\mathcal{B}$  of  $X$ , every open cover  $\mathcal{U} \subseteq \mathcal{B}$  of  $X$  has a finite subcover.

Proof. The nontrivial direction is  $\Leftarrow$ . Let  $\mathcal{B}$  be a basis and  $\mathcal{U}$  an open cover. Then by above,  $\exists$  open cover  $\mathcal{V} \subseteq \mathcal{B}$  which refines  $\mathcal{U}$ . By the hypothesis,  $\mathcal{V}$  admits a finite subcover  $\mathcal{V}_0$ , which is hence a finite refinement of  $\mathcal{U}$ .  $\square$

Examples. (a) All finite top. spaces are compact.

(b) In the cofinite top on  $\mathbb{N}$ ,  $\mathbb{N}$  is compact. In fact, every subset of  $\mathbb{N}$  is compact (in the subspace top). HW

(c)  $\mathbb{R}$  is not compact:  $\{(n-1, n+1)\}_{n \in \mathbb{Z}}$  is an open cover with no finite subcover. Also,  $\{(-n, n)\}_{n \in \mathbb{N}}$  is an open cover with no finite subcover.

The last example generalizes:

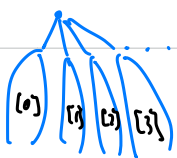
Prop. If a top. space  $X$  admits an unbounded compatible metric then  $X$  is not compact.

Proof. If  $d$  is an unbd metric on  $X$  generating the top., then for any  $x_0 \in X$ , the collection  $\{B_n^d(x_0)\}_{n \in \mathbb{N}}$  is an open cover of  $X$  but doesn't have a finite subcover.  $\square$

Cor. Compact subsets of metric spaces are bounded.

However boundedness of the metric doesn't guarantee compactness even for complete metric spaces.

Example. The Baire space  $\mathbb{N}^{\mathbb{N}}$  has compatible complete metric bdd by 1 (the usual metric). Yet  $\mathbb{N}^{\mathbb{N}}$  is not compact: the collection of cylinders  $\{[n]\}_{n \in \mathbb{N}}$  is an open cover with no finite subcover.



The noncompactness occurred because the tree was infinitely branching, in contrast to:

Example. The Cantor space  $2^{\mathbb{N}}$  is compact. In fact,  $\Sigma^{\mathbb{N}}$  is compact for finite  $\Sigma$ .

Proof. We show for  $2^{\mathbb{N}}$  but the proof is the same for any finite  $\Sigma$ .

Let  $\mathcal{U}$  be an open cover of  $2^{\mathbb{N}}$  and suppose towards a contradiction that it has no finite subcover. Call a vertex  $w \in 2^{<\mathbb{N}}$  **heavy** if  $[w]$  still doesn't have a finite subcover of  $\mathcal{U}$ . By our assumption the empty word (the root) is heavy. Note that if  $w$  is heavy then at least one of  $w0$  and  $w1$  is heavy (since otherwise if  $\mathcal{U}_0, \mathcal{U}_1 \subseteq \mathcal{U}$  are finite covers of  $[w0]$  and  $[w1]$ , then  $\mathcal{U}_0 \cup \mathcal{U}_1$  is a finite cover of  $[w]$ ). Using this we start from the root and obtain an infinite branch  $x \in 2^{\mathbb{N}}$  s.t.  $\forall n \in \mathbb{N}$ ,  $x|_n$  is heavy, i.e.  $[x|_n]$  doesn't have a finite cover in  $\mathcal{U}$ . But  $\mathcal{U}$  covers  $x$ , so  $\exists U \in \mathcal{U}$  with  $x \in U$ , so for a large enough  $n \in \mathbb{N}$ ,  $[x|_n] \subseteq U$ , but then  $\{U\}$  is an open cover of  $[x|_n]$  of size 1, a contradiction.  $\square$

## Compactness of subspaces.

Remark on the compactness of subspaces. Let  $X$  be a top. space and  $Y \subseteq X$ . An open cover of  $Y$ , by definition, is a collection  $\mathcal{U}$  of sets  $U \subseteq Y$  which are relatively open in  $Y$ , i.e.  $\exists$  set  $\tilde{U} \subseteq X$  open in  $X$  s.t.  $U = \tilde{U} \cap Y$ . For example,  $\{[0, \frac{2}{3}), (\frac{1}{2}, 1]\}$  is an open cover of  $[0, 1]$  in the subspace top. However, we could replace  $\mathcal{U}$  with the collection  $\tilde{\mathcal{U}} := \{\tilde{U} : U \in \mathcal{U}\}$ , so it consists of sets open in  $X$  and it still covers  $Y$  in the sense that  $Y \subseteq \bigcup_{\tilde{U} \in \tilde{\mathcal{U}}} \tilde{U} := \bigcup_{U \in \mathcal{U}} U$ . Clearly,  $\mathcal{U}$  has a finite subcover of  $Y \iff \tilde{\mathcal{U}}$  has a finite subcover of  $Y$ . Thus:

Obs. For a top. space  $X$ , a subspace  $Y \subseteq X$  is compact if every cover  $\mathcal{V}$  of  $Y$  (i.e.  $Y \subseteq \mathcal{V}$ ) with sets **open in  $X$** , has a finite subcover  $\mathcal{V}_0 \subseteq \mathcal{V}$  of  $Y$  (i.e.  $Y \subseteq \bigcup \mathcal{V}_0$ ).